

# Coulomb blockade of Majorana fermion induced transport

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(Dated: October 12, 2011)

We study Coulomb charging effects for transport through a topologically nontrivial superconducting wire, where Majorana bound states are present at the interface to normal-conducting leads. We construct the general Keldysh functional integral representation, and provide detailed results for the nonlinear current-voltage relation under weak Coulomb blockade conditions.

PACS numbers: 71.10.Pm, 73.23.-b, 74.50.+r

## I. INTRODUCTION

Charge transport through topologically nontrivial materials is of great current interest, offering novel fundamental insights as well as potential applications in topological quantum computing.<sup>1</sup> The recently discovered topological insulators as well as topological superconductors (TSs)<sup>2,3</sup> are predicted to exhibit spectacular nonlocal transport phenomena and have consequently attracted a lot of attention. In particular, for one-dimensional TS quantum wires, the crucial role of Majorana bound states (MBSs) located near the interface to topologically trivial regions has been emphasized.<sup>4-8</sup> Majorana fermions are special in that they are their own antiparticles, i.e., the fermion creation operator is equal to the annihilation operator, and the condensed-matter setup discussed below could offer experimental signatures for these elusive and hitherto unobserved particles.

We here study nonlinear electronic transport through a mesoscopic TS wire (also referred to as “dot”) containing a pair of MBSs. The wire is assumed to be contacted by conventional normal-conducting leads. The contacts correspond to thin insulating layers or barriers and will be modeled by a tunnel Hamiltonian. We include Coulomb interactions on the dot via the charging energy  $E_c = e^2/(2C)$ . The schematic setup studied in this work is shown in Fig. 1, and we briefly outline possible experimental realizations below. The noninteracting ( $E_c = 0$ ) version of this setup was studied in detail before,<sup>7,9-12</sup> and one finds *resonant Andreev reflection* with the maximal (unitary) value  $G_{L/R} = 2e^2/h$  of the linear conductance through the left/right contact. For the interacting case ( $E_c > 0$ ), an important work by Fu<sup>13</sup> has provided a general framework to address transport in this setup. The Hilbert space constraint introduced in Ref. 13 is, however, difficult to handle in actual calculations. In fact, the concrete results presented in Ref. 13 (see also below) were mostly restricted to the strong Coulomb blockade (CB) regime realized for large  $E_c$ . We formulate an alternative Keldysh functional integral approach below, where the constraint is automatically satisfied. We give detailed results for the weak CB (small  $E_c$ ) regime, where we find a pronounced interaction ( $E_c \neq 0$ ) induced suppression

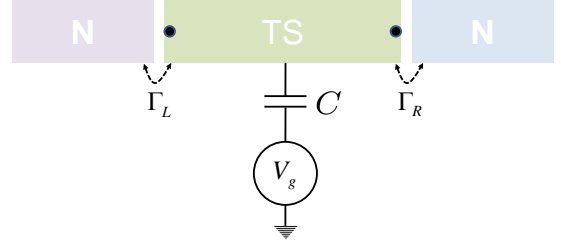


Figure 1: (Color online) Setup studied in this work: a topological superconductor (TS) dot containing Majorana bound states at the ends (indicated by black dots) is coupled via tunnel contacts (hybridizations  $\Gamma_{L,R}$ ) to normal-conducting (N) metallic electrodes. The charging energy  $E_c = e^2/(2C)$  causes phase fluctuations on the TS dot and influences the current-voltage relation. We assume that the superconducting gap  $\Delta$  of the TS is large compared to the other energy scales, i.e., quasiparticle excitations on the dot can be neglected.

of the current.

Experimental realizations for the setup shown in Fig. 1 arise under several distinct physical scenarios. For instance, a nanowire made of a 3D topological insulator (e.g.,  $\text{Bi}_2\text{Se}_3$ ) with proximity-induced superconducting correlations is predicted to have MBSs at both ends.<sup>14</sup> Similar setups can be realized in 2D topological insulators with “quantum spin Hall” edge states.<sup>13,15</sup> Another example is based on the vortex core states in  $p$ -wave superconductors<sup>9</sup> or in 2D noncentrosymmetric superconductors.<sup>16</sup> Finally, the MBS should also be realizable in semiconductor quantum wires with a Zeeman field and proximity-induced superconductivity.<sup>17</sup> The stability of the MBS when including interactions in an isolated wire has also been addressed in several recent works.<sup>18</sup> However, so far no experimental signature for the MBS has been reported in any system.

The structure of this paper is as follows. In Sec. II we describe the model employed in our study and briefly show that our formulation reproduces known results<sup>13</sup> in the strong CB limit. We then construct a general Keldysh functional integral representation for the interacting problem in Sec. III. This theory is employed in Sec. IV to evaluate the current-voltage characteristics un-

der weak CB conditions. We briefly summarize our main findings and provide an outlook in Sec. V. Technical details and derivations can be found in three appendices. Note that we often use units where  $\hbar = k_B = 1$ .

## II. MODEL

We consider transport through a mesoscopic TS quantum wire contacted by normal-conducting leads, see Fig. 1. We assume throughout this work that the superconducting gap  $\Delta$  is the largest relevant energy scale, and thus no quasiparticles above the gap are involved. In particular,  $\max(eV, E_c) < \Delta$ . We study a “floating” (not grounded) TS dot, where current must be conserved under steady-state conditions. In some previous works, see, e.g., Ref. 9, a grounded superconductor was considered, where current need not be conserved. Note also that the situation is different for conventional (topologically trivial) superconducting dots, where transport is typically mediated by cotunneling processes and the resulting conductance is much smaller.<sup>19</sup> In the present case, two MBSs may host a fermion at no energy cost, and thus no even-odd parity effect<sup>19</sup> arises.

### A. Model

For realistical TS wire length, the two MBSs located near the left/right ( $j = L/R$ ) contact correspond to *de-coupled* Majorana fermion operators,  $\gamma_j = \gamma_j^\dagger$ , with anticommutator relation  $\{\gamma_i, \gamma_j\} = \delta_{ij}$ . In the low-energy sector of interest here, only the MBSs provide available fermion states to tunnel through the superconductor. It is convenient to introduce the complex auxiliary fermion  $d = (\gamma_L + i\gamma_R)/\sqrt{2}$ , i.e.,

$$\gamma_L = (d + d^\dagger)/\sqrt{2}, \quad \gamma_R = -i(d - d^\dagger)/\sqrt{2}. \quad (1)$$

For an isolated wire, the fermion parity  $(-)^n$  of the two degenerate ground states with even ( $n = 2N$ ) or odd ( $n = 2N + 1$ ) number of electrons fixes the occupation of the  $d$  fermion level,

$$2i\gamma_L\gamma_R = 2\hat{n}_d - 1 = (-)^{n+1}, \quad \hat{n}_d = d^\dagger d. \quad (2)$$

This imposes a constraint on the Hilbert space,<sup>13</sup> and the Majorana fermion dynamics cannot be taken as independent of the charge (or the dual phase) dynamics. Below, we instead formulate an alternative but essentially equivalent approach free of any Hilbert space constraint, see also Ref. 20, which is technically easier to handle. To that end, consider the number operator  $\hat{N}$  for Cooper pairs conjugate to the condensate phase  $\chi$  on the dot,  $[\chi, \hat{N}] = i$ . The dot’s instantaneous charge state is then fully determined by specifying the configuration  $(N, n_d)$ , where  $N \in \mathbb{N}_0$  and  $n_d = 0, 1$  are eigenvalues of  $\hat{N}$  and  $\hat{n}_d$ , respectively. Importantly, Eq. (2) tells us that the

fermion parity of the dot is determined by  $n_d$  only and not affected by changes of  $N$ . By construction, both  $\hat{N}$  and  $\chi$  commute with  $\gamma_j$ , and therefore no constraint on the Hilbert space arises.

The full Hamiltonian describing transport through the dot,  $H = H_c + H_t + H_l$ , contains the Coulomb charging term

$$H_c = E_c(2\hat{N} + \hat{n}_d - n_0)^2, \quad (3)$$

where  $n_0 \in \mathbb{R}$  is tunable by the backgate voltage  $V_g$  indicated in Fig. 1. Electrons in the leads correspond to effectively spinless fermion operators  $c_{j=L/R,k}$  with momentum  $k$ , where for given  $j$ ,  $c_{jk}$  and  $c_{jk}^\dagger$  are tunnel-coupled to the respective Majorana fermion  $\gamma_j$  only. For simplicity, we assume a  $k$ -independent tunnel matrix element  $\lambda_j$  encoding the overlap of the lead state with the respective MBS wavefunction. As detailed in App. A, the tunnel Hamiltonian then takes the form

$$H_t = \frac{1}{\sqrt{2}} \sum_k \left[ \lambda_L c_{L,k}^\dagger (d + e^{-i\chi} d^\dagger) - i\lambda_R c_{R,k}^\dagger (d - e^{-i\chi} d^\dagger) \right] + \text{h.c.}, \quad (4)$$

which is charge conserving. In particular, the different terms in  $H_t$  describe electron tunneling from the dot to lead  $j$ , either by destroying the  $d$  state without changing  $N$ , i.e.,  $(N, 1) \rightarrow (N, 0)$ , or by occupying the  $d$  state together with a splitting of a Cooper pair, i.e.,  $(N, 0) \rightarrow (N + 1, 1)$ , plus the conjugate processes. As elaborated in Appendix B, we treat the lead Hamiltonian  $H_l$  within the standard wide-band approximation.<sup>21,22</sup> The applied bias voltage  $V$  corresponds to the chemical potential difference in both leads,  $eV = \mu_L - \mu_R$ . For further reference, we also define the hybridization energy scales

$$\Gamma_j = 2\pi\nu_j|\lambda_j|^2, \quad (5)$$

where  $\nu_j$  is the density of states in lead  $j = L/R$ .

It is now straightforward to rederive the well-known results<sup>9–12</sup> for the noninteracting ( $E_c = 0$ ) limit from the above Hamiltonian, see also Sec. IV. Before turning to the construction of the Keldysh functional integral for the interacting problem in Sec. III, let us briefly show that the above formulation easily reproduces previous results<sup>13</sup> reporting electron “teleportation” under strong CB conditions, i.e., for  $\max(eV, \Gamma_{L,R}) \ll E_c$ .

### B. Strong Coulomb Blockade

Under strong CB conditions, the charging energy (3) is dominant and needs to be examined first. We focus on the resonant situation realized for half-integer values for  $n_0$ , where two different cases arise: (i) For  $n_0 = 2\ell + 1/2$  with integer  $\ell$ , degeneracy is achieved for fixed Cooper pair number  $N = \ell$ , where the  $d$  state is either occupied

( $n_d = 1$ ) or empty ( $n_d = 0$ ). (ii) When  $n_0 = 2\ell - 1/2$ , the degeneracy point is achieved for different Cooper pair numbers, namely for  $(N, n_d) = (\ell - 1, 1)$  and  $(\ell, 0)$ . We conclude that the low-energy description needs to keep only two states, either of type (i) or (ii) depending on  $n_0$ . For case (i),  $N$  remains constant and all terms  $\propto e^{\pm i\chi}$  (where  $N$  is raised or lowered by one unit) can be omitted in Eq. (4). The tunnel Hamiltonian then takes the form

$$H_t^{(i)} = \frac{1}{\sqrt{2}} \sum_k \left( \lambda_L c_{L,k}^\dagger - i\lambda_R c_{R,k}^\dagger \right) d + \text{h.c.}$$

Under scenario (ii) only transitions  $(\ell - 1, 1) \leftrightarrow (\ell, 0)$  are possible. Using  $f = e^{-i\chi} d^\dagger$  as effective single-charge fermion, we arrive at

$$H_t^{(ii)} = \frac{1}{\sqrt{2}} \sum_k \left( \lambda_L c_{L,k}^\dagger + i\lambda_R c_{R,k}^\dagger \right) f + \text{h.c.}$$

In both cases, we recover the resonant tunneling model describing teleportation.<sup>13</sup> This leads to the unitary conductance value  $G_L = G_R = e^2/h$  instead of the resonant Andreev reflection value  $2e^2/h$  found when  $E_c = 0$ .

### III. KELDYSH FUNCTIONAL INTEGRAL

We now turn to the general Keldysh functional integral<sup>21,22</sup> for the interacting problem. This corresponds to the Majorana generalization of the (real-time version of the) Ambegaokar-Eckern-Schön (AES) action for a conventional metallic grain.<sup>23–25</sup>

#### A. AES action for Majorana induced transport

The relevant low-energy degree of freedom characterizing the interacting problem is the single-electron (half Cooper pair) counting phase field  $\phi(t) \equiv \chi/2$ . For the

nonequilibrium problem at hand, we employ the textbook Keldysh functional integral formulation<sup>21,22</sup> and double the phase field  $\phi(t) \rightarrow \phi_\pm(t)$  according to the forward and backward branch of the Keldysh time contour. Technically, the phase field is introduced through a Hubbard-Stratonovich transformation of the charging energy in Eq. (3), see App. B for details. Physically,  $\phi$  is the dual variable to the Cooper pair charge  $N$ . We then introduce classical/quantum phase fields,<sup>24,25</sup>

$$\phi_c(t) = \frac{1}{2}(\phi_+ + \phi_-), \quad \phi_q(t) = \phi_+ - \phi_-. \quad (6)$$

The charging energy (3) yields the action contribution

$$S_c = \int dt \dot{\phi}_q \left( \frac{\dot{\phi}_c}{2E_c} + n_0 \right) \quad (7)$$

encoding Coulomb blockade effects. By virtue of the Hubbard-Stratonovich transformation, the fields corresponding to the lead fermions and to the Majorana fermions effectively become noninteracting. In the functional integral, those fields can therefore be integrated out analytically; for technical details, see Appendix B. The final (phase representation of the) functional integral for the Keldysh partition function reads

$$\mathcal{Z} = \int \mathcal{D}(\phi_c, \phi_q) e^{i(S_c + S_f)}. \quad (8)$$

The fermion field integrations result in the action piece

$$S_f = -i \sum_{j=L/R} \ln \text{Pf} [\tilde{\tau}_x i \partial_t \delta(t - t') + \tilde{\Lambda}_j(t, t')], \quad (9)$$

where “Pf” denotes the Pfaffian and  $\tilde{\tau}_{x,y,z}$  are Pauli matrices in (rotated<sup>22</sup>) Keldysh space. Moreover, we define the antisymmetric (time and Keldysh) matrix

$$\begin{aligned} \tilde{\Lambda}_j(t, t') = & 2i\Gamma_j F(t - t') \begin{pmatrix} c(t)c(t') \cos \Phi_j(t, t') & c(t)s(t') \sin \Phi_j(t, t') \\ -s(t)c(t') \sin \Phi_j(t, t') & s(t)s(t') \cos \Phi_j(t, t') \end{pmatrix} \\ & + i\Gamma_j \begin{pmatrix} 0 & c^2(t)\delta_-(t - t') - s^2(t)\delta_+(t - t') \\ -c^2(t)\delta_+(t - t') + s^2(t)\delta_-(t - t') & 0 \end{pmatrix} \end{aligned} \quad (10)$$

with  $c(t) = \cos[\phi_q(t)/2]$ ,  $s(t) = \sin[\phi_q(t)/2]$ , and  $\delta_\pm(t) \equiv \delta(t \pm 0^+)$ , where the infinitesimal shifts reflect the proper causality features. At temperature  $T$ , the distribution function  $F(t - t')$  has the Fourier transform

$$F(\epsilon) = \tanh[\epsilon/(2T)], \quad (11)$$

and the phase function  $\Phi_j$  appearing in Eq. (10) is

$$\Phi_j(t, t') = \mu_j(t - t') + \phi_c(t) - \phi_c(t'). \quad (12)$$

Finally, the hybridizations  $\Gamma_j$  were defined in Eq. (5).

While Eq. (8), with the action in Eqs. (7) and (9), provides a general representation of the interacting nonequilibrium Majorana transport problem, approximations are necessary in order to obtain concrete analytical results.

## B. Semiclassical expansion

We here are mostly interested in the weak CB regime, where standard arguments<sup>24</sup> imply that a semiclassical approximation in the phase representation is appropriate. In particular, fluctuations of  $\phi_q$  around zero are small,

$$\begin{aligned}\check{\Lambda}_j^{(0)} &= i\Gamma_j \begin{pmatrix} 2F(t-t')\cos\Phi_j(t,t') & \delta_-(t-t') \\ -\delta_+(t-t') & 0 \end{pmatrix}, \\ \check{\Lambda}_j^{(1)} &= i\Gamma_j F(t-t')\sin\Phi_j(t,t') \begin{pmatrix} 0 & \phi_q(t') \\ -\phi_q(t) & 0 \end{pmatrix}, \\ \check{\Lambda}_j^{(2)} &= -i\frac{\Gamma_j}{4}F(t-t')\cos\Phi_j(t,t') \begin{pmatrix} \phi_q^2(t) + \phi_q^2(t') & 0 \\ 0 & -2\phi_q(t)\phi_q(t') \end{pmatrix} + \frac{\Gamma_j}{2}\phi_q^2(t)\delta(t-t')\check{\tau}_y.\end{aligned}$$

At this stage, it is convenient to introduce the *interacting* Keldysh-Majorana Green's function (GF)

$$\check{G}_j(t, t') = \left( \check{\tau}_x i\partial_t + \check{\Lambda}_j^{(0)} \right)^{-1}. \quad (13)$$

This has a triangular representation in (rotated) Keldysh space,  $\check{G}_j = \begin{pmatrix} 0 & G_j^A \\ G_j^R & G_j^K \end{pmatrix}$ , with retarded/advanced GFs  $G_j^{R/A}$  and Keldysh GF  $G_j^K$ . Note that in this representation, the Majorana GF has a bosonic Keldysh structure, see also Ref. 12. According to Eq. (13),  $G_j^{R/A}$  obeys the noninteracting ( $\phi_c = 0$ ) equation of motion and thus has the Fourier representation  $G_j^{R/A}(\epsilon) = 1/(\epsilon \pm i\Gamma_j)$ . This implies the spectral function

$$A_j(\epsilon) = -\text{Im} [G_j^R(\epsilon)] = \frac{\Gamma_j}{\epsilon^2 + \Gamma_j^2}. \quad (14)$$

i.e., we can expand the action in Eq. (8) to quadratic order in  $\phi_q$ . Keeping the full nonlinear  $\phi_c$  dependence, the matrix (10) is thus approximated by  $\check{\Lambda}_{j=L/R} \simeq \check{\Lambda}_j^{(0)} + \check{\Lambda}_j^{(1)} + \check{\Lambda}_j^{(2)}$ . Specifically, the matrices  $\check{\Lambda}_j^{(m)}(t, t')$  of order  $\phi_q^m$  are

Only  $G_j^K$  is affected by interactions encoded in  $\phi_c$ .

The action (9) now takes the form

$$S_f = -i \sum_j \ln \text{Pf}(\check{G}_j^{-1}) + S_f^{(1)} + S_f^{(2)} + O(\phi_q^3), \quad (15)$$

where the first ( $\phi_q = 0$ ) term vanishes as a result of the unitary time evolution along the closed Keldysh contour. The first-order (in  $\phi_q$ ) contribution is

$$S_f^{(1)} = -\frac{i}{2} \sum_j \text{Tr}(\check{G}_j \check{\Lambda}_j^{(1)}) = \int dt \mathcal{I} \phi_q, \quad (16)$$

where the trace “Tr” extends both over time and Keldysh space, and  $\mathcal{I}$  denotes the total current flowing into the dot. Some algebra yields

$$\mathcal{I}(t) = \sum_j \Gamma_j \int dt' G_j^R(t-t') \{F_{j,s}(t'-t) \cos[\phi_c(t') - \phi_c(t)] + F_{j,a}(t'-t) \sin[\phi_c(t') - \phi_c(t)]\} \quad (17)$$

with the time-symmetric (-antisymmetric) functions

$$F_{j,s}(t) = F(t) \sin(\mu_j t), \quad F_{j,a}(t) = F(t) \cos(\mu_j t). \quad (18)$$

Finally, the second-order term in Eq. (15) reads

$$S_f^{(2)} = -\frac{i}{2} \sum_j \text{Tr} \left[ \check{G}_j \check{\Lambda}_j^{(2)} - \frac{1}{2} \left( \check{G}_j \check{\Lambda}_j^{(1)} \right)^2 \right]. \quad (19)$$

## IV. WEAK COULOMB BLOCKADE LIMIT

### A. Langevin approach

When the charging energy  $E_c$  is sufficiently small, fluctuations of the phase fields  $\phi_{c,q}$  are small and allow for a quadratic expansion of the action  $S_f$  in *both* fields.<sup>24</sup> This expansion is detailed in Appendix C and allows to reformulate the functional integral (8) in terms of the

equivalent semiclassical Langevin equation,

$$\frac{1}{2E_c} \ddot{\phi}_c(t) + \int^t dt' \eta(t-t') \dot{\phi}_c(t') = \xi(t). \quad (20)$$

The *damping kernel*  $\eta(t-t')$  has the Fourier representation

$$\eta(\omega) = \sum_{j=L/R} \frac{\Gamma_j}{\omega} \int \frac{d\epsilon}{2\pi} (A_j(\epsilon + \omega/2) + A_j(\epsilon - \omega/2)) [F_{j,a}(\epsilon + \omega/2) - F_{j,a}(\epsilon - \omega/2)]. \quad (21)$$

The Gaussian noise field  $\xi(t)$  has zero mean value and the correlation function  $\langle \xi(t)\xi(t') \rangle = K(t-t')$ , where the *fluctuation kernel* has the Fourier representation

$$K(\omega) = \sum_j \frac{\Gamma_j}{2} \int \frac{d\epsilon}{2\pi} (A_j(\epsilon + \omega/2) + A_j(\epsilon - \omega/2)) [1 - F_{j,a}(\epsilon + \omega/2)F_{j,a}(\epsilon - \omega/2)] \\ - \sum_j \Gamma_j^2 \int \frac{d\epsilon}{2\pi} \text{Re} [G_j^R(\epsilon + \omega/2)G_j^R(\epsilon - \omega/2)] F_{j,s}(\epsilon + \omega/2)F_{j,s}(\epsilon - \omega/2). \quad (22)$$

In equilibrium ( $\mu_{L,R} = 0$ ),  $F_{j,s} = 0$  and  $F_{j,a} = F$ , and hence the fluctuation-dissipation relation

$$K_{\text{eq}}(\omega) = \frac{\omega}{2} \coth\left(\frac{\omega}{2T}\right) \eta_{\text{eq}}(\omega) \quad (23)$$

is satisfied, providing an important consistency check.

In the zero-temperature limit, both  $\eta(\omega)$  and  $K(\omega)$  can be evaluated in closed form. Specifically, we find for the

damping kernel

$$\eta_{T=0}(\omega) = \sum_{j,\pm} \frac{\Gamma_j}{\pi\omega} \tan^{-1}\left(\frac{\omega \pm \mu_j}{\Gamma_j}\right), \quad (24)$$

while the fluctuation kernel is

$$K_{T=0}(\omega) = \frac{|\omega|\eta(\omega)}{2} + \sum_j \frac{\Gamma_j}{2\pi} \Theta(2|\mu_j| - |\omega|) \left[ \tan^{-1}\left(\frac{|\mu_j|}{\Gamma_j}\right) + \tan^{-1}\left(\frac{|\mu_j| - |\omega|}{\Gamma_j}\right) + \frac{\Gamma_j}{|\omega|} \ln \frac{(|\mu_j| - |\omega|)^2 + \Gamma_j^2}{\mu_j^2 + \Gamma_j^2} \right], \quad (25)$$

where  $\Theta$  is the Heaviside function.

The Langevin equation (20) allows for an intuitive interpretation of the charge dynamics in this interacting Majorana system in terms of an effective  $RC$  circuit. Its solution is used below in order to compute the current-voltage relation.

## B. Current-voltage relation

Next we outline the calculation of the dc current,  $I \equiv I_L = -I_R$ . By using Eq. (17) and the noise average  $\langle \dots \rangle_\xi = \int \mathcal{D}\xi (\dots) e^{-\frac{1}{2}\xi K^{-1}\xi}$ , we find the (indeed  $t$ -independent) result

$$I_j = \Gamma_j \int dt' G_j^R(t-t') [F_{j,s}(t'-t) \langle \cos[\bar{\phi}_c(t') - \bar{\phi}_c(t)] \rangle_\xi \\ + F_{j,a}(t'-t) \langle \sin[\bar{\phi}_c(t') - \bar{\phi}_c(t)] \rangle_\xi].$$

$F_{j,a/s}$  is given in Eq. (18), and

$$\bar{\phi}_c(t) = \int^t d\tau D^R(t-\tau) \xi(\tau) \quad (26)$$

solves Eq. (20) for a given noise trajectory, where  $D^R(t-\tau)$  is the retarded solution of Eq. (20) with  $\xi(t) \rightarrow \delta(t-\tau)$ . Performing the Gaussian noise average, we obtain

$$I_j = \Gamma_j \int dt' G_j^R(t-t') F_{j,s}(t'-t) e^{-J(t-t')}, \quad (27)$$

where the noise correlations are encoded in  $J(t) = J(-t)$ , similar to the standard “ $P(E)$ ” theory of dynamical CB,<sup>26</sup>

$$J(t-t') = \frac{1}{2} \left\langle [\bar{\phi}_c(t) - \bar{\phi}_c(t')]^2 \right\rangle_\xi. \quad (28)$$

Note that in order to obtain  $J(t-t')$  one first has to solve the Langevin equation (20) for a given noise trajectory, and then average over all noise realizations.

Since  $J(t - t') \geq 0$ , interactions always decrease the current (27) with respect to the noninteracting solution,  $I_j^{(0)}$ . The latter follows by setting  $J \rightarrow 0$  in Eq. (27). In fact, after Fourier transformation, we get the known result<sup>7,9-12</sup>

$$I_j^{(0)} = \Gamma_j \int \frac{d\epsilon}{2\pi} F(\epsilon - \mu_j) A_j(\epsilon) \quad (29)$$

with the Lorentzian spectral function in Eq. (14). In linear response ( $\mu_j \rightarrow 0$ ) and taking the zero-temperature limit, we recover resonant Andreev reflection with quantized conductance  $G_j = e\partial I_j/\partial \mu_j = 2e^2/h$ .

Equation (27) for the current-voltage relation in the weak Coulomb blockade regime is one of the main results of this work. We use this expression later on to quantify the effect of Coulomb interactions ( $E_c \neq 0$ ) on the transport properties in this Majorana system.

### C. Retardation effects

The Langevin equation (20) in general contains memory effects because of the frequency-dependence of the damping kernel. These are most pronounced for  $T = 0$ , where  $\eta(\omega)$  takes the form in Eq. (24). For  $\omega \rightarrow 0$ , this gives the finite value

$$\eta_0 = \frac{2}{\pi} \sum_j \frac{1}{1 + (\mu_j/\Gamma_j)^2}, \quad (30)$$

while for  $|\omega| \rightarrow \infty$ , we have  $\eta(\omega) \simeq \Gamma/|\omega|$  with  $\Gamma = \Gamma_L + \Gamma_R$ . On low frequency scales,  $|\omega| < \min(\sqrt{\mu_j^2 + \Gamma_j^2})$ , the kernel  $\eta(\omega)$  can be approximated by  $\eta(\omega) \simeq \eta_0$ . This approximation corresponds to the absence of retardation in the damping kernel  $\eta(t)$ . This approximation works even better for finite  $T$ , where

$$\eta_0 = \sum_j \frac{\Gamma_j}{T} \int \frac{d\epsilon}{2\pi} \frac{1}{\cosh^2(\frac{\epsilon}{2T})} \frac{\Gamma_j}{(\epsilon - \mu_j)^2 + \Gamma_j^2} \quad (31)$$

yields a damping suppression,  $\eta_0 \simeq \Gamma/(2T)$ , in the high- $T$  limit.

We therefore employ the simplification  $\eta(\omega) = \eta_0$  from now on, which is appropriate to describe low-energy transport. This implies from Eq. (20) that we now have a Markovian Langevin equation,

$$\ddot{\phi}_c(t) + \Omega \dot{\phi}_c(t) = 2E_c \xi(t), \quad (32)$$

and  $D^R(t)$  in Eq. (26) simplifies to

$$D^R(t) = \frac{1}{\eta_0} (1 - e^{-\Omega t}) \Theta(t). \quad (33)$$

The inverse “ $RC$  time” in Eq. (32) is given by

$$\Omega = (RC)^{-1} = \eta_0 E_c. \quad (34)$$

Since  $\eta_0 \leq 4/\pi$ , see Eq. (30), and using  $E_c = e^2/(2C)$ , the effective resistance  $R$  in Eq. (34) always fulfills  $R \geq h/(4e^2)$ . Note that this corresponds to a parallel resistance as seen by the dot.

However, at the same time it is essential to keep the full frequency-dependence in the fluctuation kernel  $K(\omega)$  when evaluating  $J(t)$  in Eq. (28). Using Eq. (33), some algebra yields

$$J(t) = \frac{1}{\pi\eta_0^2} \int_0^\infty d\omega K(\omega) \frac{1 - \cos(\omega t)}{\omega^2(1 + \omega^2/\Omega^2)}, \quad (35)$$

where the current  $I = I_L$  follows from Eq. (27).

### D. Zero-bias anomaly

We now present results for the  $IV$  relation in the zero-temperature limit, where MBS effects are most pronounced. We adopt the Markovian approximation described in Sec. IV C and study a symmetric system, where  $\Gamma_L = \Gamma_R = \Gamma/2$  and  $\mu_L = -\mu_R = eV/2$ . As relevant transport quantity, we define the dimensionless nonlinear conductance

$$g(V) = \frac{I(V)}{e^2 V/h}. \quad (36)$$

For  $E_c = 0$ , the exact solution [Eq. (29)] yields

$$g^{(0)}(V) = \frac{\Gamma}{eV} \tan^{-1} \left( \frac{eV}{\Gamma} \right). \quad (37)$$

Note that the unitary limit (resonant Andreev reflection) corresponds to  $g = 1$ , which is reached for  $V \rightarrow 0$  in Eq. (37).

We now write  $J(t) = J_0 + J_1$ , where

$$J_0(t) = \frac{1}{2\pi\eta_0} \int_0^\infty \frac{d\omega}{\omega} \frac{1 - \cos(\omega t)}{1 + \omega^2/\Omega^2} \quad (38)$$

comes from the first term ( $\propto \eta(\omega) = \eta_0$ ) in Eq. (25).  $J_1$  is a pure nonequilibrium term: the relevant kernel contribution scales as  $\sim V^3$ . Let us then first consider the *linear response regime*, where  $J_1$  can be discarded. To logarithmic accuracy,  $J_0(t) \simeq (2\pi\eta_0)^{-1} \ln(\Omega t)$  for  $\Omega t > 1$ , where  $\eta_0 = 4/\pi$  within linear response, see Eq. (30). Equation (27) then yields the dimensionless linear conductance in analytical form,

$$g(V \rightarrow 0) = c_0 (E_c/\Gamma)^{-1/8}, \quad c_0 \approx 0.96. \quad (39)$$

This result holds<sup>27</sup> for  $\Gamma \lesssim E_c$ . As a function of  $E_c/\Gamma$ , Eq. (39) reveals a *power-law suppression* of the linear conductance with the universal exponent  $1/8$ , reminiscent of a zero-bias anomaly. For arbitrary  $E_c/\Gamma$ , the current given by Eq. (27) can be evaluated numerically. The resulting linear conductance is shown in Fig. 2, and the analytical result (39) is quite accurate in describing the

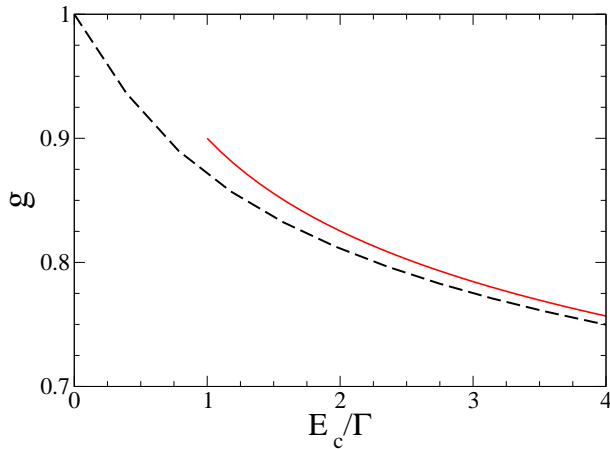


Figure 2: (Color online) Dimensionless linear conductance,  $g = g(V \rightarrow 0)$ , vs  $E_c/\Gamma$  in the weak CB regime. The black dashed curve has been obtained from numerical analysis of Eq. (27), with  $J(t)$  in Eq. (35) and  $K(\omega)$  in Eq. (25). The red solid curve gives the analytical result in Eq. (39) valid for  $E_c \gtrsim \Gamma$ .

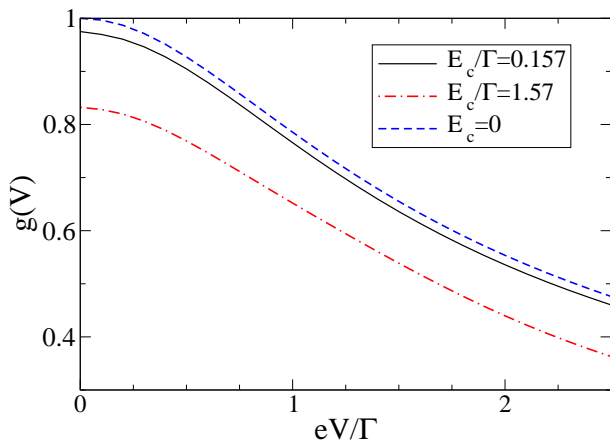


Figure 3: (Color online) Dimensionless nonlinear conductance  $g(V)$  vs  $eV/\Gamma$  for several values of the charging energy. Results for  $E_c > 0$  were obtained by numerical integration, the result for  $E_c = 0$  is exact.

interaction ( $E_c \neq 0$ ) induced suppression of the current for  $E_c \gtrsim \Gamma$ .

Next we turn to the nonlinear conductance  $g(V)$  in Eq. (36). Numerical results for several values of  $E_c/\Gamma$  are shown in Fig. 3, where the exact result [Eq. (37)] for  $E_c = 0$  is also depicted. We find a clear suppression of  $g(V)$  when the applied bias voltage increases. This suppression becomes stronger with increasing  $E_c$ . In fact, we observe that the interaction induced suppression of the current is more pronounced for small voltage. We therefore interpret our results as a zero-bias anomaly caused by the (weak) Coulomb blockade of Majorana fermion induced transport.

## V. CONCLUDING REMARKS

In this work, we have presented a general theory of transport through a mesoscopic superconductor containing a pair of MBSs. We have studied the regime of low-energy transport, where metallic leads are attached and all relevant energy scales are small compared to the superconducting gap. The case of weak Coulomb blockade, where the charging energy  $E_c$  does not significantly exceed the typical hybridization scale  $\Gamma$ , has been investigated in some detail, and we found clear signatures for an interaction ( $E_c \neq 0$ ) induced suppression of both the linear and the nonlinear conductance.

We have introduced a formulation to analyze interaction effects in electronic transport through MBSs which does not rely on the Hilbert space constraints of previous approaches and thus is easier to handle in general. We are presently extending this formulation to study several other situations of current interest. In particular, relevant questions include the full counting statistics and the nonequilibrium dephasing in multi-terminal transport geometries containing MBSs, and the Josephson effect with superconducting leads. Furthermore, the strong Coulomb blockade regime also allows for the detailed study of the full Keldysh functional integral (8) in terms of a rate equation approach.

To conclude, we are confident that the characteristic features reported here, i.e., the suppression of the linear conductance with increasing  $E_c/\Gamma$  and of the nonlinear conductance with  $V/\Gamma$  in the limit of weak Coulomb blockade, can be observed experimentally once MBSs have been detected. The setup studied here may allow to find signatures of the elusive Majorana fermions in transport properties, including phenomena such as resonant Andreev reflection, teleportation, and the universal power-law suppression of the linear conductance,  $g \sim (E_c/\Gamma)^{-1/8}$ , for intermediate values of the ratio  $E_c/\Gamma$ .

## Acknowledgments

This work was supported by the SFB TR 12 of the DFG and by the Spanish MICINN under contract FIS2008-04209.

## Appendix A: Derivation of the tunnel Hamiltonian

In order to keep the paper self-contained, we here provide a derivation of the tunnel Hamiltonian (4). Following Flensberg,<sup>11</sup> the field operator  $\Psi_\sigma(x)$  for electrons with spin projection  $\sigma = \uparrow, \downarrow$  in a TS wire of length  $L$  (with  $0 < x < L$ ) defines the Majorana fermion operators

$$\gamma_{j=L/R} = \sum_\sigma \int_0^L dx [f_{j\sigma}^*(x)\Psi_\sigma(x) + f_{j\sigma}(x)\Psi_\sigma^\dagger(x)]$$

with the MBS wavefunction  $f_{j\sigma}(x)$ . Projecting the full Nambu spinor onto the MBS subspace, one obtains<sup>11</sup>

$$\Psi_\sigma(x) \rightarrow \sum_j f_{j\sigma}(x) \gamma_j.$$

The tunnel Hamiltonian now reads

$$\begin{aligned} H_t &= \sum_{jk\sigma} \int_0^L dx t_j^*(x) C_{jk\sigma}^\dagger \Psi_\sigma(x) + \text{h.c.} \\ &= \sum_{jk\sigma} V_{j\sigma}^* C_{jk\sigma}^\dagger \gamma_j + \text{h.c.}, \end{aligned}$$

where  $C_{jk\sigma}^\dagger$  creates an electron in lead  $j = L/R$  with momentum  $k$  and spin projection  $\sigma$ . Here  $t_j(x)$  is assumed  $k$ -independent for simplicity, which defines the tunnel matrix elements  $V_{j\sigma} = \int dx t_j(x) f_{j\sigma}^*(x)$ . Note that one can always form suitable linear combinations of  $C_{jk,\uparrow}$  and  $C_{jk,\downarrow}$  to form spinless lead fermions  $c_{jk}$  coupled to the MBSs,

$$\sum_\sigma V_{j\sigma}^* C_{jk\sigma}^\dagger \rightarrow \lambda_j c_{jk}^\dagger.$$

The other (orthogonal) linear combination then decouples from the problem. Finally, inserting the auxiliary fermion representation (1) and taking into account charge conservation, we arrive at the tunnel Hamiltonian quoted in Eq. (4).

## Appendix B: On the Keldysh functional integral

Here we provide a detailed derivation of the AES action in Sec. III A. We start by constructing the Lagrangian  $L_c$  for the isolated dot in terms of  $\phi(t)$  and the Grassmann variable  $d(t)$ , plus the corresponding “velocities”  $\dot{\phi} = \frac{1}{2} \partial H_c / \partial N$  and  $\dot{d}$ . Noting that for Grassmann variables,  $i\dot{d}$  is canonically dual to  $d$ , we have

$$L_c = \frac{\dot{\phi}^2}{4E_c} + n_0 \dot{\phi} + \bar{d}(i\partial_t - \dot{\phi})d.$$

Adding the tunnel contribution and performing the gauge transformation  $d \rightarrow e^{-i\phi}d$ , we obtain

$$\begin{aligned} L_c + L_t &= \frac{\dot{\phi}^2}{4E_c} + n_0 \dot{\phi} + \frac{i}{2} \sum_j \gamma_j \dot{\gamma}_j \\ &\quad - \sum_{jk} (\lambda_j \bar{\psi}_{jk} e^{-i\phi} \gamma_j + \text{h.c.}). \end{aligned}$$

Here we switched back from the auxiliary  $d$  fermion to the Majorana field  $\gamma_j$ . Note that up to a full time derivative,  $i\dot{d} \rightarrow (i/2) \sum_j \gamma_j \dot{\gamma}_j$ . The Grassmann fields  $(\psi_{jk}, \bar{\psi}_{jk})$  correspond to the lead fermion operators  $(c_{jk}, c_{jk}^\dagger)$ . Using the Keldysh formulation, we double all fields according to the forward and backward branch of the Keldysh time contour, i.e.,  $\gamma_j(t) \rightarrow (\gamma_{j,+}(t), \gamma_{j,-}(t))^T$  and so on. It is also convenient to gauge out the chemical potentials  $\mu_j$  in the leads,  $\psi_{jk,\pm}(t) \rightarrow e^{i\mu_j t} \psi_{jk,\pm}(t)$ . We now use the Keldysh matrix notation  $\check{\phi}(t) = \phi_c + \tilde{\tau}_z \phi_q / 2$ , see Eq. (6). The complete Keldysh action,  $S = S_\gamma + S_c + S_t + S_l$ , contains  $S_c$  in Eq. (7) and the pieces

$$\begin{aligned} S_\gamma &= \frac{i}{2} \sum_j \int dt \gamma_j \tilde{\tau}_z \dot{\gamma}_j, \\ S_t &= - \sum_{jk} \lambda_j \int dt \bar{\psi}_{jk} \tilde{\tau}_z e^{-i[\mu_j t + \check{\phi}(t)]} \gamma_j + \text{h.c.}, \\ S_l &= \sum_{jk} \int dt \bar{\psi}_{jk} \tilde{\tau}_z (i\partial_t - \epsilon_{jk}) \psi_{jk}, \end{aligned}$$

where  $\epsilon_{jk}$  refers to the dispersion relation in lead  $j$ .

The current  $I_j$  flowing from lead  $j$  into the dot follows from the Heisenberg equation of motion. The currents obey the relation  $(I_L + I_R)(t) = \delta S_t / \delta \phi_q(t)$ . Current conservation implies  $\langle I_L \rangle = -\langle I_R \rangle$ , which fixes the mean value  $\langle \dot{\phi}_c \rangle$  to the chemical potential  $\mu_s$  of the superconducting dot. The latter has to be determined self-consistently from current conservation. We then redefine  $\phi_c \rightarrow \mu_s t + \phi_c(t)$ , i.e.,  $\phi_c(t)$  now refers to fluctuations around the mean-field value  $(\mu_s t)$  of the classical phase variable. The ensuing changes in  $S_c$  and  $S_t$  can be absorbed in a renormalization of  $n_0$  and  $\mu_j$ . In particular, denoting their bare values by  $\bar{n}_0$  and  $\bar{\mu}_j$ , respectively, we find  $n_0 = \bar{n}_0 + \mu_s / (2E_c)$  and  $\mu_j = \bar{\mu}_j + \mu_s$ .

The next step is to integrate out the Grassmann fields  $(\psi_{jk}, \bar{\psi}_{jk})$ . This is a standard step<sup>21,22,24,25</sup> and leads to the effective Majorana action  $S_{\text{eff}}$  replacing  $S_l + S_t + S_\gamma$ . Before turning to the result, we perform the usual rotation in Keldysh space in order to have triangular Green’s function (GF) representations, using the unitary matrix<sup>21</sup>  $\check{L} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ . The rotated Majorana fields are  $\tilde{\gamma}_j = \check{L} \gamma_j$ . Evaluating the resulting momentum integrals for the leads in wide-band approximation, we find the effective Majorana action



$$S_{\text{eff}} = \frac{1}{2} \sum_j \int dt dt' \tilde{\gamma}_j^T(t) [\tilde{\tau}_x i \partial_t \delta(t-t') + \check{Q}_j(t, t')] \tilde{\gamma}_j(t')$$

$$\check{Q}_j(t, t') = i\Gamma_j e^{i\tilde{\tau}_x \phi_q(t)/2} \begin{pmatrix} \delta_-(t-t') & 2F(t-t')e^{i\Phi_j(t, t')} \\ 0 & -\delta_+(t-t') \end{pmatrix} \tilde{\tau}_x e^{-i\tilde{\tau}_x \phi_q(t')/2}.$$

The hybridizations  $\Gamma_j$  are defined in Eq. (5) with lead density of states  $\nu_j = \sum_k \delta(\epsilon_{jk})$ ,  $F(t)$  has the Fourier transform (11), and the phase function  $\Phi_j$  is specified in Eq. (12).

Now we are ready to also integrate out the Majorana fields. This finally yields Eq. (8) where

$$\check{\Lambda}_j(t, t') = \frac{1}{2} [\check{Q}_j(t, t') - \check{Q}_j^T(t', t)]$$

is the antisymmetric part of  $\check{Q}_j$ .

### Appendix C: Derivation of Langevin equation

Here we provide some details concerning the derivation of the Langevin equation (20) in Sec. IV A. We first analyze the action piece  $S_f^{(1)} = \int dt \mathcal{I} \phi_q$ . As mentioned in App. B, the chemical potential  $\mu_s$  of the dot has to be chosen self-consistently in order to fulfill current conservation. By construction, we must therefore have the average value  $\mathcal{I} = 0$  when  $\dot{\phi}_c = 0$ . Using Eq. (17), expanding to first order in  $\phi_c$ , and performing a partial integration, we find

$$S_f^{(1)} = - \int dt \phi_q(t) \int dt' \eta(t-t') \dot{\phi}_c(t'),$$

where  $\eta(t-t')$  is determined by Eq. (21). Next we note that  $S_f^{(2)}$ , which is already of order  $\phi_q^2$ , can be evaluated by replacing the interacting Keldysh GF  $G_j^K$  with its noninteracting ( $\phi_c = 0$ ) version. After some algebra, we find from Eq. (19)

$$S_f^{(2)} = \frac{i}{2} \int dt dt' \phi_q(t) K(t-t') \phi_q(t')$$

with  $K(t-t')$  determined by Eq. (22). The Fourier-transformed kernel  $K(\omega)$  directly describes finite-frequency current noise correlations for  $E_c = 0$ , a question that has been studied for  $\omega \rightarrow 0$  in Ref. 12. Finally, we complete the derivation by Hubbard-Stratonovich transformation of the  $\phi_q^2$  term in the generating functional [Eq. (8)],

$$e^{iS_f^{(2)}} = \int \mathcal{D}\xi \, e^{-\frac{i}{2} \int dt dt' \xi(t) K^{-1}(t-t') \xi(t') + i \int dt \phi_q \xi}.$$

Functional integration over  $\phi_q$ , which appears only linearly in the action, yields Eq. (20), and the Gaussian noise field  $\xi(t)$  indeed has the correlation function  $K(t-t')$  in Eq. (22).

- <sup>1</sup> C. Nayak, S.H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. **80**, 1083 (2008).
- <sup>2</sup> M.Z. Hasan and C.L. Kane, Rev. Mod. Phys. **82**, 3045 (2010).
- <sup>3</sup> X.L. Qi and S.C. Zhang, Rev. Mod. Phys. (in press); arXiv:1008.2026.
- <sup>4</sup> A.Yu. Kitaev, Phys. Usp. **44**, 131 (2001).
- <sup>5</sup> L. Fu and C.L. Kane, Phys. Rev. Lett. **100**, 096407 (2008).
- <sup>6</sup> Y. Tanaka, T. Yokoyama, and N. Nagaosa, Phys. Rev. Lett. **103**, 107002 (2009).
- <sup>7</sup> K.T. Law, P.A. Lee, and T.K. Ng, Phys. Rev. Lett. **103**, 237001 (2009).
- <sup>8</sup> J. Linder, Y. Tanaka, T. Yokoyama, A. Sudbø, and N. Nagaosa, Phys. Rev. Lett. **104**, 067001 (2010).
- <sup>9</sup> C.J. Bolech and E. Demler, Phys. Rev. Lett. **98**, 237002 (2007).
- <sup>10</sup> J. Nilsson, A.R. Akhmerov, and C.W.J. Beenakker, Phys. Rev. Lett. **101**, 120403 (2008).
- <sup>11</sup> K. Flensberg, Phys. Rev. B **82**, 180516(R) (2010).
- <sup>12</sup> A. Golub and B. Horovitz, Phys. Rev. B **83**, 153415 (2011).

- <sup>13</sup> L. Fu, Phys. Rev. Lett. **104**, 056402 (2010).
- <sup>14</sup> M. Cook and M. Franz, arXiv:1105.1787.
- <sup>15</sup> L. Fu and C.L. Kane, Phys. Rev. B **79**, 161408(R) (2009).
- <sup>16</sup> M. Sato and S. Fujimoto, Phys. Rev. B **79**, 094504 (2009).
- <sup>17</sup> R.M. Lutchyn, J.D. Sau, and S. Das Sarma, Phys. Rev. Lett. **105**, 077001 (2010); Y. Oreg, G. Refael, and F.v. Oppen, *ibid.* **105**, 177002 (2010).
- <sup>18</sup> S. Gangadharaiah, B. Braunecker, P. Simon, and D. Loss, Phys. Rev. Lett. **107**, 036801 (2011); E.M. Stoudenmire, J. Alicea, O.A. Starykh, and M.P.A. Fisher, Phys. Rev. B **84**, 014503 (2011); E. Sela, A. Altland, and A. Rosch, Phys. Rev. B **84**, 085114 (2011).
- <sup>19</sup> D.V. Averin and Yu.V. Nazarov, Phys. Rev. Lett. **69**, 1993 (1992).
- <sup>20</sup> B. van Heck, F. Hassler, A.R. Akhmerov, and C.W.J. Beenakker, arXiv:1108.1095.
- <sup>21</sup> Yu.V. Nazarov and Ya.M. Blanter, *Quantum transport* (Cambridge University Press, Cambridge, 2009).
- <sup>22</sup> A. Altland and B.D. Simons, *Condensed matter field theory*, 2nd edition (Cambridge University Press, Cambridge,

- 2010).
- <sup>23</sup> G. Schön and A.D. Zaikin, Phys. Rep. **198**, 237 (1990).
- <sup>24</sup> Yu.V. Nazarov, Phys. Rev. Lett. **82**, 1245 (1999); D.S. Golubev and A.D. Zaikin, Phys. Rev. Lett. **86**, 4887 (2001); Phys. Rev. B **69**, 075318 (2004); P.W. Brouwer, A. Lamacraft, and K. Flensberg, Phys. Rev. B **72**, 075316 (2005).
- <sup>25</sup> A. Altland and R. Egger, Phys. Rev. Lett. **102**, 026805 (2009).
- <sup>26</sup> H. Grabert and M.H. Devoret (eds.), *Single Charge Tunneling*, NATO ASI Series B, Physics v.294 (Plenum Press, New York, 1992).
- <sup>27</sup> The weak CB regime is eventually left for  $\Gamma \ll E_c$ , where Eq. (39) ceases to be valid.